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Approximate Solutions of a System of Differential Equations[1]

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1. INTRODUCTION

McEwen [3] considered the problem of approximating the solution of the linear differential equation

$$L(y) \equiv y^{(n)}(t) + f_1(t)y^{(n-1)}(t) + \ldots + f_n(t)y(t) = r(t)$$

subject to the nonhomogeneous two-point boundary conditions

$$U_i(y) \equiv \sum_{j=0}^{n-1} \{\alpha_{ij}y^{(j)}(a) + \beta_{ij}y^{(j)}(b)\} = h_i, \qquad i = 1, 2, \ldots, n,$$

where the functions $f_1(t), f_2(t), \ldots, f_n(t)$, and $r(t)$ are defined and continuous on the interval $a \leqslant t \leqslant b$ and the n boundary conditions are linearly independent. He showed that if this system has a unique solution, then for every $m \geqslant n$ there exists a unique polynomial $p_m(t)$ of degree at most m which satisfies the boundary conditions $U_i(p_m) = h_i$, $i = 1, 2, \ldots, n$, and which best approximates the solution of the system in the sense that the integral

$$\int_b^a |L(p_m) - r(t)|^p \, dt, \qquad p \geqslant 1 \text{ and fixed,}$$

is a minimum.

In this paper we extend McEwen's work and consider the system of equations

$$\mathbf{L}(\mathbf{y}(t)) \equiv [\mathbf{D} + \mathbf{F}(t)]\,\mathbf{y}(t) = \mathbf{r}(t) \tag{1.1}$$

subject to the linearly independent boundary conditions

$$\mathbf{U}(\mathbf{y}) \equiv \mathbf{A}\mathbf{y}(a) + \mathbf{B}\mathbf{y}(b) = \mathbf{h}, \tag{1.2}$$

where $\mathbf{D}$ is the n by n diagonal operator matrix $[d/dt]$, $\mathbf{F}(t) = [f_{ij}(t)]$ is an n by n matrix of functions continuous on the interval $a \leqslant t \leqslant b$, $\mathbf{y}(t)$ is an n-dimensional column vector, $\mathbf{A} = [\alpha_{ij}]$ and $\mathbf{B} = [\beta_{ij}]$ are n by n matrices of constants, and $\mathbf{h} = [h_i]$ is an n-dimensional vector of constants. The linear independence of the boundary conditions implies that the matrix $[\mathbf{A}; \mathbf{B}]$ is of

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rank n. By introducing auxiliary functions, the nth order linear equation considered by McEwen can be written equivalently as a system of n simultaneous first-order linear differential equations (see, for example, [4]). This system of equations can be written in the form (1.1) by introducing matrix notation. It is therefore included in our discussion. We consider the existence, uniqueness, and convergence of sequences of vectors $\mathbf{p}_m(t)$ of polynomials of degree at most m which satisfy the boundary conditions (1.2) and which best approximate the solution of the system (1.1) and (1.2) in the sense that they minimize the integral

$$\int_a^b |\mathbf{L}(\mathbf{p}(t) - \mathbf{r}(t)|^p \, dt, \qquad p \geqslant 1 \text{ and fixed,}$$

over the class of vectors $\mathbf{p}(t)$ of this form, where, for any matrix $\mathbf{A} = [a_{ij}]$, $|\mathbf{A}|$ is defined by the equation

$$|\mathbf{A}| = \sum_{i,j} |a_{ij}|.$$

A discussion of the matrix norm $|\mathbf{A}|$ can be found in [4].

2. The Solution of a System of Linear Differential Equations

It is known [1] that the system (1.1) and (1.2) has a unique solution if, and only if, the homogeneous system

$$\mathbf{L}(\mathbf{y}) = \mathbf{0}, \qquad \mathbf{U}(\mathbf{y}) = \mathbf{0}$$

has no nontrivial solution. Barrett and Jacobson [1] have expressed the solution of the system (1.1) and (1.2) in terms of a Green's function for the system. The Green's function, $\mathbf{G}(t,s)$, for the system is an n by n matrix defined as follows:

(i) The columns of $\mathbf{G}(t,s)$ satisfy the homogeneous system except at $t = s$, so that

$$[\mathbf{D} + \mathbf{F}(t)]\,\mathbf{G}(t,s) = \mathbf{0}, \qquad t \neq s,$$

and

(ii) The Green's function $\mathbf{G}(t,s)$ is continuous in t except for the diagonal elements, each of which has a single discontinuity at $t = s$ in such a way that

$$\mathbf{G}(s^+, s) - \mathbf{G}(s^-, s) = \mathbf{I},$$

where $\mathbf{I}$ is the n by n identity matrix.

The Green's function exists and is unique if, and only if, the homogeneous system has no nontrivial solution.

Let $\mathbf{C}$ be a nonsingular submatrix of $[\mathbf{A};\mathbf{B}]$ having column indices $k_1, k_2, \ldots, k_n$. We can write

$$\mathbf{C} = [\mathbf{A};\mathbf{B}]\,\mathbf{M},$$

where the only nonzero elements of the $2n$ by n matrix $\mathbf{M}$ occur in its n by n submatrix having row indices k_1, k_2, ..., k_n, and this submatrix is the n by n identity matrix. Then the following formula expresses the solution of the system (1.1) and (1.2) in terms of the Green's function.

THEOREM 1. *Assume that the homogeneous system corresponding to the system* (1.1) *and* (1.2) *has no nontrivial solution. Then the system* (1.1) *and* (1.2) *has the unique solution*

$$\mathbf{y}(t) = \int_a^b \mathbf{G}(t,s)\,\mathbf{r}(s)\,ds + [\mathbf{G}(t,a); -\mathbf{G}(t,b)]\,\mathbf{M}\mathbf{C}^{-1}\,\mathbf{h}.$$

Note that this theorem also holds when $\mathbf{r}(t) = \mathbf{0}$ or $\mathbf{h} = \mathbf{0}$.

3. EXISTENCE AND UNIQUENESS OF APPROXIMATING VECTORS OF POLYNOMIALS

Our first theorem establishes the existence of vectors of polynomials which best approximate the solution of the system (1.1) and (1.2) in the sense described in the introduction.

THEOREM 2. *Assume that the system of differential equations* (1.1) *and* (1.2) *has a unique solution. Let $p \geqslant 1$ be given. Then for every positive integer m there exists a vector $\mathbf{p}_m(t)$ of polynomials of degree at most m which satisfies the boundary conditions* (1.2) *and which minimizes the integral*

$$\int_a^b |\mathbf{L}(\mathbf{p}(t)) - \mathbf{r}(t)|^p\,dt \tag{3.1}$$

over the class of vectors of polynomials of this kind.

The proof of this theorem depends on the following lemma, which is a vector generalization of Lemma I of [*2*].

LEMMA 1. *Let $\mathbf{g}_1(t)$, $\mathbf{g}_2(t)$, ..., $\mathbf{g}_d(t)$ be n-dimensional vector functions of t, continuous and linearly independent on the interval $a \leqslant t \leqslant b$, and let*

$$\mathbf{\Phi}(t) = c_1\,\mathbf{g}_1(t) + c_2\,\mathbf{g}_2(t) + \ldots + c_d\,\mathbf{g}_d(t)$$

be an arbitrary linear combination of these vector functions. Let M denote the maximum of $|\mathbf{\Phi}(t)|$ on the interval $a \leqslant t \leqslant b$. Then there is a constant Q, depending only on the vector functions $\mathbf{g}_i(t)$, $i = 1, 2, \ldots, d$, and on the interval $a \leqslant t \leqslant b$, such that

$$|c_i| \leqslant QM, \qquad i = 1, 2, \ldots, d.$$

Proof of Lemma 1. For each value of $k = 1, 2, \ldots, d$, determine the constants in the expression

$$\mathbf{\Phi}_k(t) = c_{k,1}\,\mathbf{g}_1(t) + c_{k,2}\,\mathbf{g}_2(t) + \ldots + c_{k,d}\,\mathbf{g}_d(t)$$

so that

$$\int_a^b \mathbf{\Phi}_k{}^T(t)\,\mathbf{g}_i(t)\,dt = \delta_{ik}, \qquad i = 1, 2, \ldots, d,$$

where the superscript T denotes the transpose of the vector and where δ_{ik} is the Kronecker delta. This amounts to requiring that the constants $c_{k,i}$, $i = 1, 2, \ldots, d$, satisfy the d simultaneous equations

$$\begin{aligned}
c_{k,1}\int_a^b \mathbf{g}_1{}^T\mathbf{g}_1\,dt + c_{k,2}\int_a^b \mathbf{g}_2{}^T\mathbf{g}_1\,dt + \ldots + c_{k,d}\int_a^b \mathbf{g}_d{}^T\mathbf{g}_1\,dt &= \delta_{1k}\\
c_{k,1}\int_a^b \mathbf{g}_1{}^T\mathbf{g}_2\,dt + c_{k,2}\int_a^b \mathbf{g}_2{}^T\mathbf{g}_2\,dt + \ldots + c_{k,d}\int_a^b \mathbf{g}_d{}^T\mathbf{g}_2\,dt &= \delta_{2k}\\
&\vdots\\
c_{k,1}\int_a^b \mathbf{g}_1{}^T\mathbf{g}_d\,dt + c_{k,2}\int_a^b \mathbf{g}_2{}^T\mathbf{g}_d\,dt + \ldots + c_{k,d}\int_a^b \mathbf{g}_d{}^T\mathbf{g}_d\,dt &= \delta_{dk}.
\end{aligned}$$

Considering the constants $c_{k,i}$, $i = 1, 2, \ldots, d$, as unknowns in this system, suppose that the determinant of their coefficients were zero. Then a set of constants, not all zero, could be determined for a vector of functions

$$\mathbf{\Phi}_0(t) = c_{01}\,\mathbf{g}_1(t) + c_{02}\,\mathbf{g}_2(t) + \ldots + c_{0d}\,\mathbf{g}_d(t)$$

so as to make

$$\int_a^b \mathbf{\Phi}_0{}^T(t)\,\mathbf{g}_i(t)\,dt = 0, \qquad i = 1, 2, \ldots, d.$$

Then we would have

$$\int_a^b \mathbf{\Phi}_0{}^T(t)\mathbf{\Phi}_0(t)\,dt = c_{01}\int_a^b \mathbf{\Phi}_0{}^T(t)\,\mathbf{g}_1(t)\,dt + c_{02}\int_a^b \mathbf{\Phi}_0{}^T(t)\,\mathbf{g}_2(t)\,dt + \ldots + c_{0d}\int_a^b \mathbf{\Phi}_0{}^T(t)\,\mathbf{g}_d(t)\,dt = 0.$$

Since $\mathbf{\Phi}_0{}^T(t)\mathbf{\Phi}_0(t)$ is the sum of the squares of the components of $\mathbf{\Phi}_0(t)$, the vanishing of this integral would imply that $\mathbf{\Phi}_0(t)$ is identically the zero vector. This would contradict the linear independence of the $\mathbf{g}_i(t)$. It follows that the determinant of the coefficients of the c_{ik} is not zero, so these constants are uniquely determined.

Choose Q' so that $Q' \geqslant |\mathbf{\Phi}_k(t)|$ for $a \leqslant t \leqslant b$ and for $k = 1, 2, \ldots, d$. Then

$$\left|\int_a^b \mathbf{\Phi}_k{}^T(t)\,\mathbf{\Phi}(t)\,dt\right| \leqslant \int_a^b \left|\mathbf{\Phi}_k{}^T(t)\right| \cdot \left|\mathbf{\Phi}(t)\right| dt \leqslant (b-a)\,Q'\,M.$$

By construction, the left member of this inequality is $|c_k|$. It follows that

$$|c_k| \leqslant QM, \qquad k = 1, 2, \ldots, d,$$

where $Q = Q'(b-a)$.

Proof of Theorem 2. Any n-dimensional vector $\mathbf{p}(t)$ of polynomials of degree at most m can be written as

$$\mathbf{p}(t) = \begin{bmatrix} c_{10} + c_{11}t + \ldots + c_{1m}t^m \\ c_{20} + c_{21}t + \ldots + c_{2m}t^m \\ \ldots \\ c_{n0} + c_{n1}t + \ldots + c_{nm}t^m \end{bmatrix}$$

$$= \begin{bmatrix} c_{10} & c_{11} & \ldots & c_{1m} \\ c_{20} & c_{21} & \ldots & c_{2m} \\ \ldots \\ c_{n0} & c_{n1} & \ldots & c_{nm} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^m \end{bmatrix} = \mathbf{C}\boldsymbol{\tau}(t),$$

where $\boldsymbol{\tau}(t)$ is the $m+1$ dimensional column vector $\{1, t, t^2, \ldots, t^m\}$. The condition that $\mathbf{p}(t)$ satisfies the boundary condition (1.2) is that

$$\mathbf{AC}\boldsymbol{\tau}(a) + \mathbf{BC}\boldsymbol{\tau}(b) = \mathbf{h}.$$

When this equation is written as n linear equations involving the $n(m+1)$ elements of $\mathbf{C}$ as unknowns, it is seen that when $m \geqslant 1$ there is an infinite number of matrices $\mathbf{C}$ which satisfy this equation. The general solution matrix can be written as the sum of a particular solution $\mathbf{J}_0$ and an arbitrary linear combination of a basis of fewer than $n(m+1)$, say $\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_d$, solutions of the associated homogeneous equation

$$\mathbf{AC}\boldsymbol{\tau}(a) + \mathbf{BC}\boldsymbol{\tau}(b) = \mathbf{0}.$$

Then the vectors $\mathbf{p}(t)$ of polynomials of degree at most m which satisfy the boundary condition (1.2) are those of the form

$$\mathbf{p}(t) = (\mathbf{J}_0 + c_1\mathbf{J}_1 + c_2\mathbf{J}_2 + \ldots + c_d\mathbf{J}_d)\boldsymbol{\tau}(t),$$

where the constants $c_1, c_2, \ldots, c_d$ are arbitrary.

The problem is to determine a vector of polynomials from this class which minimizes the integral (3.1). Since this integral is nonnegative, it has a nonnegative greatest lower bound δ_m, and there must be a sequence of polynomials

$$\mathbf{p}(j;t) = (\mathbf{J}_0 + c_{1j}\mathbf{J}_1 + \ldots + c_{dj}\mathbf{J}_d)\boldsymbol{\tau}(t), \qquad j = 1, 2, \ldots,$$

such that

$$\lim_{j\to\infty} \int_a^b |\mathbf{L}(\mathbf{p}(j;t)) - \mathbf{r}(t)|^p\,dt = \delta_m.$$

Consequently there is a number j_0 such that for $j > j_0$,

$$\int_a^b |\mathbf{L}(\mathbf{p}(j;t)) - \mathbf{r}(t)|^p\,dt < \delta_m + 1.$$

By Hölder's inequality,

$$\int_a^b |\mathbf{L}(\mathbf{p}(j;t)) - \mathbf{r}(t)|^p\,dt \geqslant (b-a)^{1-p}\left[\int_a^b |\mathbf{L}(\mathbf{p}(j;t)) - \mathbf{r}(t)|\,dt\right]^p,$$

so for $j > j_0$,

$$(b-a)^{(p-1)/p}(\delta_m+1)^{1/p} \geqslant \int_a^b |\mathbf{L}(\mathbf{p}(j;s)) - \mathbf{r}(s)|\, ds$$
$$\geqslant \int_a^t |\mathbf{L}(p(j;s))|\, ds - \int_a^t |\mathbf{r}(s)|\, ds$$

for $a \leqslant t \leqslant b$. It follows that for $j > j_0$,

$$(b-a)^{(p-1)/p}(\delta_m+1)^{1/p} + \int_a^b |\mathbf{r}(s)|\, ds \geqslant \int_a^t |\mathbf{L}(\mathbf{p}(j;s))|\, ds$$
$$\geqslant \left| \sum_{i=1}^{d} c_{ij} \int_a^t \mathbf{L}[\mathbf{J}_i \boldsymbol{\tau}(s)]\, ds \right| - \sup_{a \leqslant t \leqslant b} \left| \int_a^t \mathbf{L}[\mathbf{J}_0 \boldsymbol{\tau}(s)]\, ds \right|.$$

Define the constant M by

$$M = (b-a)^{(p-1)/p}(\delta_m+1)^{1/p} + \int_a^b |\mathbf{r}(s)| ds + \sup_{a \leqslant t \leqslant b} \left| \int_a^t \mathbf{L}[\mathbf{J}_0 \boldsymbol{\tau}(s)]\, ds \right|.$$

Then for $j > j_0$,

$$M \geqslant \left| \sum_{i=1}^{d} c_{ij} \int_a^t \mathbf{L}[\mathbf{J}_i \boldsymbol{\tau}(s)]\, ds \right|.$$

The vectors of functions

$$\mathbf{g}_i(t) = \int_a^t \mathbf{L}[\mathbf{J}_i \boldsymbol{\tau}(s)]\, ds, \qquad a \leqslant t \leqslant b, \qquad i = 1, 2, \ldots, d,$$

are linearly independent. Otherwise, there would be constants k_i, not all zero, such that

$$\sum_{i=1}^{d} k_i \int_a^t \mathbf{L}[\mathbf{J}_i \boldsymbol{\tau}(s)]\, ds \equiv \mathbf{0}.$$

By differentiation we would have

$$\sum_{i=1}^{d} k_i \mathbf{L}[\mathbf{J}_i \boldsymbol{\tau}(t)] \equiv \mathbf{L}\left[\sum_{i=1}^{d} k_i \mathbf{J}_i \boldsymbol{\tau}(t)\right] \equiv \mathbf{0}.$$

Since the matrices $\mathbf{J}_i$ are linearly independent by construction, the vector of polynomials

$$\mathbf{p}(t) = \sum_{i=1}^{d} k_i \mathbf{J}_i \boldsymbol{\tau}(t)$$

would not vanish identically. By the construction of the matrices $\mathbf{J}_i$, this sum would satisfy the homogeneous boundary condition

$$\mathbf{A}\mathbf{p}(a) + \mathbf{B}\mathbf{p}(b) = \mathbf{0}.$$

Therefore, $\mathbf{p}(t)$ would be a nontrivial solution of the homogeneous system

$$\mathbf{L}(\mathbf{y}) \equiv \mathbf{0}, \qquad \mathbf{U}(\mathbf{y}) = \mathbf{0}.$$

Since this would contradict the hypothesis that the system (1.1) and (1.2) has a unique solution, the vectors of functions $\mathbf{g}_i(t)$ must be linearly independent.

It now follows by Lemma 1 that there is a constant Q such that

$$|c_{ij}| \leqslant MQ$$

for $i = 1, 2, \ldots, d$ and $j > j_0$. The sequence $(c_{1j}, c_{2j}, \ldots, c_{dj})$ therefore lies in a bounded region of d-dimensional space, and necessarily has a finite limit point $(c_1, c_2, \ldots, c_d)$. This point determines the vector of polynomials

$$\mathbf{p}_m(t) = [\mathbf{J}_0 + c_1 \mathbf{J}_1 + c_2 \mathbf{J}_2 + \ldots + c_d \mathbf{J}_d]\boldsymbol{\tau}(t)$$

of degree at most m which satisfies the boundary condition (1.2). Since the value of the integral (3.1) depends continuously on the coefficients of $\mathbf{J}_1, \mathbf{J}_2, \ldots, \mathbf{J}_d$, the vector $\mathbf{p}_m(t)$ minimizes the value of the integral for polynomials of this kind. The conclusion follows.

The approximating vector of polynomials of Theorem 2 is not necessarily unique. For fixed m, let $\mathscr{P}_m$ denote the set of vectors of polynomials $\mathbf{p}_m(t)$ which satisfy the conclusion of Theorem 2. The properties of this set are easily described in terms of the linear space $L_p^n[a, b]$ of all n-dimensional vectors $\mathbf{f}(t)$ of functions such that

$$\|\mathbf{f}(t)\| = \left[\int_a^b |\mathbf{f}(t)|^p\, dt\right]^{1/p} < \infty.$$

The functional $\|\mathbf{f}\|$ is a norm on $L_p^n[a,b]$, and the space is complete in the metric determined by this norm.

THEOREM 3. *The set $\mathscr{P}_m$ is contained in a finite dimensional subspace of $L_p^n[a,b]$ and is closed, bounded, and convex.*

Proof of Theorem 3. In the notation used in the proof of Theorem 2, every element of $\mathscr{P}_m$ is a linear combination of the vectors $\mathbf{J}_0\boldsymbol{\tau}(t), \mathbf{J}_1\boldsymbol{\tau}(t), \ldots, \mathbf{J}_d\boldsymbol{\tau}(t)$. Therefore $\mathscr{P}_m$ is contained in a finite dimensional subspace of $L_p^n[a,b]$. The continuous dependence of the integral (3.1) on the coefficients in this linear combination shows that the set $\mathscr{P}_m$ is closed. If a linear combination of these vectors belongs to $\mathscr{P}_m$, then from the proof of Theorem 2, the coefficients are bounded. It follows that $\mathscr{P}_m$ is bounded.

Suppose that $\mathbf{p}_m(0;t)$ and $\mathbf{p}_m(1;t)$ are two vectors of polynomials which belong to $\mathscr{P}_m$, and let

$$\delta_m = \int_a^b |\mathbf{L}(\mathbf{p}_m(i;t)) - \mathbf{r}(t)|^p\, dt, \qquad i = 0, 1.$$

Then for $0 \leqslant \lambda \leqslant 1$, the vector of polynomials

$$\mathbf{p}_m(\lambda;t) = \lambda\mathbf{p}_m(1;t) + (1-\lambda)\,\mathbf{p}_m(0;t)$$

satisfies the boundary conditions (1.2). By Minkowski's inequality,

$$\left\{\int_a^b |\mathbf{L}(\mathbf{p}_m(\lambda;t)) - \mathbf{r}(t)|^p\,dt\right\}^{1/p}$$
$$\leqslant \lambda\left\{\int_a^b |\mathbf{L}(\mathbf{p}_m(1;t)) - \mathbf{r}(t)|^p\,dt\right\}^{1/p} + (1-\lambda)\left\{\int_a^b |\mathbf{L}(\mathbf{p}_m(0;t) - \mathbf{r}(t)|^p\,dt\right\}^{1/p}$$
$$= \delta_m^{1/p}.$$

Since $\mathbf{p}_m(0;t)$ and $\mathbf{p}_m(1;t)$ are minimizing vectors of polynomials, equality must hold and we must have

$$\int_a^b |\mathbf{L}(\mathbf{p}_m(\lambda;t)) - \mathbf{r}(t)|^p\,dt = \delta_m.$$

Therefore, for $0 \leqslant \lambda \leqslant 1$, $\mathbf{p}_m(\lambda;t)$ belongs to $\mathscr{P}_m$, and the set is convex.

Necessary and sufficient conditions that equality holds in Minkowski's inequality can easily be verified. The following corollary is a statement of these conditions in terms of the approximating vectors $\mathbf{p}_m(0;t)$ and $\mathbf{p}_m(1;t)$.

COROLLARY. *Assume that* $\mathbf{p}_m(0;t)$ *and* $\mathbf{p}_m(1;t)$ *belong to the set* $\mathscr{P}_m$. *If* $p \geqslant 1$, *then the corresponding components of the vectors*

$$\mathbf{L}(\mathbf{p}_m(0;t)) - \mathbf{r}(t) \qquad \text{and} \qquad \mathbf{L}(\mathbf{p}_m(1;t)) - \mathbf{r}(t)$$

are of the same sign for all values of t in the interval $a \leqslant t \leqslant b$. If $p > 1$, then

$$|\mathbf{L}(\mathbf{p}_m(0;t)) - \mathbf{r}(t)| = |\mathbf{L}(\mathbf{p}_m(1;t)) - \mathbf{r}(t)|$$

for all values of t in the interval $a \leqslant t \leqslant b$.

4. PRELIMINARY LEMMAS

The lemmas given in this section are needed for the discussion of convergence. The one dimensional analogues of these results are given by McEwen [3]. The first lemma is a generalization of the Weierstrass polynomial approximation theorem.

LEMMA 2. *Let* $\mathbf{f}(t)$ *be a given n-dimensional vector of functions which are defined and have continuous derivatives of order k on the interval* $a \leqslant t \leqslant b$. *Let* $\mathbf{f}(t)$ *satisfy the boundary conditions* (1.2). *Then for every* $\epsilon > 0$ *there exists a vector* $\mathbf{p}(t)$ *of polynomials which satisfies the boundary conditions* (1.2) *and such that*

$$|\mathbf{f}^{(i)}(t) - \mathbf{p}^{(i)}(t)| < \epsilon$$

for $i = 0, 1, \ldots, k$ *and* $a \leqslant t \leqslant b$.

Proof of Lemma 2. Denote the components of $\mathbf{f}(t)$ by $f_1(t), f_2(t), \ldots, f_n(t)$. By Theorem B of [3], for any $\epsilon > 0$ and for each $j = 1, 2, \ldots, n$, there is a polynomial Φ_j such that

$$|f_j^{(i)}(t) - \Phi_j^{(i)}(t)| < \epsilon/n$$

for $i = 0, 1, \ldots, k$ and $a \leqslant t \leqslant b$. Let $\mathbf{\Phi}(t)$ be the column vector $\{\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t)\}$. Then

$$|\mathbf{f}^{(i)}(t) - \mathbf{\Phi}^{(i)}(t)| < \epsilon$$

for $i = 0, 1, \ldots, k$ and $a \leqslant t \leqslant b$.

Define the constant vector $\mathbf{g}$ by

$$\mathbf{g} = \mathbf{h} - [\mathbf{A\Phi}(a) + \mathbf{B\Phi}(b)] = \mathbf{A}[\mathbf{f}(a) - \mathbf{\Phi}(a)] + \mathbf{B}[\mathbf{f}(b) - \mathbf{\Phi}(b)],$$

so that

$$|\mathbf{g}| \leqslant |\mathbf{A}| \cdot |\mathbf{f}(a) - \mathbf{\Phi}(a)| + |\mathbf{B}| \cdot |\mathbf{f}(b) - \mathbf{\Phi}(b)| < (|\mathbf{A}| + |\mathbf{B}|)\,\epsilon.$$

Determine a vector $\mathbf{q}(t)$ of linear polynomials by the conditions

$$\mathbf{Aq}(a) + \mathbf{Bq}(b) = \mathbf{g}$$

$$\mathbf{Cq}(a) + \mathbf{Dq}(b) = \mathbf{0},$$

where $\mathbf{C}$ and $\mathbf{D}$ are arbitrarily chosen n by n matrices such that the matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

is nonsingular. Since a linear polynomial is determined completely by its value at two points, the components of $\mathbf{q}(t)$ are completely determined by these conditions. Writing

$$\begin{bmatrix} \mathbf{q}(a) \\ \mathbf{q}(b) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{g} \\ \mathbf{0} \end{bmatrix},$$

we see that

$$\sum_{i=1}^{n} \{|q_i(a)| + |q_i(b)|\} \leqslant \left| \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \right| \cdot |\mathbf{g}| \leqslant K\epsilon,$$

where

$$K = (|\mathbf{A}| + |\mathbf{B}|) \left| \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \right|.$$

Since the polynomials $q_i(t)$ are linear, for $a \leqslant t \leqslant b$,

$$|q_i(t)| \leqslant |q_i(a)| + |q_i(b)|$$

and

$$q_i{}'(t) = [q_i(b) - q_i(a)]/(b - a).$$

Therefore, for $a \leqslant t \leqslant b$,

$$|\mathbf{q}(t)| \leqslant K\epsilon \qquad \text{and} \qquad |\mathbf{q}'(t)| \leqslant K\epsilon/(b-a).$$

All of the higher derivatives of $\mathbf{q}(t)$ are zero, so for $j = 0, 1, \ldots,$

$$|\mathbf{q}^{(j)}(t)| \leqslant K\epsilon, \quad a \leqslant t \leqslant b,$$

where K is a constant which depends only on the interval and the matrices **A**, **B**, **C**, and **D**.

The vector of polynomials

$$\mathbf{p}(t) = \mathbf{\Phi}(t) + \mathbf{q}(t)$$

satisfies the boundary conditions (1.2), and, for $j = 0, 1, \ldots, k$,

$$|\mathbf{f}^{(j)}(t) - \mathbf{p}^{(j)}(t)|$$
$$= |\mathbf{f}^{(j)}(t) - \mathbf{\Phi}^{(j)}(t) - \mathbf{q}^{(j)}(t)| \leqslant |\mathbf{f}^{(j)}(t) - \mathbf{\Phi}^{(j)}(t)| + |\mathbf{q}^{(j)}(t)| \leqslant (1+K)\,\epsilon,$$

for $a \leqslant t \leqslant b$. The desired conclusion follows when ϵ is replaced by $\epsilon/(1+K)$.

The next lemma is concerned with the order of approximation that may be attained with a vector of polynomials of specified degree. It is the vector generalization of Theorem D of [3].

LEMMA 3. *Under the hypotheses of Lemma* 2, *assume that* $\mathbf{f}^{(k)}(t)$ *satisfies a Lipschitz condition on the interval* $a \leqslant t \leqslant b$, i.e., *there is a constant* λ *such that*

$$|\mathbf{f}^{(k)}(t_1) - \mathbf{f}^{(k)}(t_2)| \leqslant \lambda |t_1 - t_2|$$

whenever $a \leqslant t_1, t_2 \leqslant b$. *Then for each positive integer m there exists a vector* $\mathbf{p}(t)$ *of polynomials of degree at most m which satisfies the boundary conditions* (1.2) *and such that*

$$|\mathbf{f}^{(j)}(t) - \mathbf{p}^{(j)}(t)| \leqslant B/m, \qquad j = 0, 1, 2, \ldots, k,$$

for all t in the interval $a \leqslant t \leqslant b$, *where B is a constant independent of m.*

Proof of Lemma 3. Since $\mathbf{f}^{(k)}(t)$ satisfies a Lipschitz condition, each component $f^{(k)}(t)$ also satisfies a Lipschitz condition on the interval. By Theorem D of [3], for each positive integer m there is a polynomial $q_i(t)$ of degree at most m such that

$$|f_i^{(j)}(t) - q_i^{(j)}(t)| \leqslant C/m, \qquad j = 0, 1, \ldots, k,$$

whenever $a \leqslant t \leqslant b$, where C is a constant independent of m. If $\mathbf{q}(t)$ is the column vector $\{q_1(t), q_2(t), \ldots, q_n(t)\}$, then

$$|\mathbf{f}^{(j)}(t) - \mathbf{q}^{(j)}(t)| \leqslant nC/m, \qquad j = 0, 1, \ldots, k,$$

whenever $a \leqslant t \leqslant b$. As in the proof of Lemma 2, we may construct from $\mathbf{q}(t)$ a vector $\mathbf{p}(t)$ of polynomials of degree at most m which satisfies the boundary conditions (1.2) and such that

$$|\mathbf{f}^{(j)}(t) - \mathbf{p}^{(j)}(t)| < (1 + K)nC/m, \qquad j = 0, 1, \ldots, k,$$

whenever $a \leqslant t \leqslant b$, where K is a constant independent of m. The desired conclusion follows on setting $B = (1 + K)nC$.

5. Convergence of Sequences of Approximating Vectors

For each $m = 1, 2, \ldots$, let $\mathbf{p}_m(t)$ denote a vector of polynomials of degree at most m which satisfies the conclusion of Theorem 2. In this section we consider the convergence properties of this sequence of approximating vectors.

Theorem 4. *Under the hypotheses of Theorem* 2, *let*

$$\mathbf{p}_1(t), \mathbf{p}_2(t), \ldots, \mathbf{p}_m(t), \ldots \tag{5.1}$$

be a sequence of vectors of polynomials of degree at most m which satisfy the conclusion of Theorem 2 *for some fixed* $p \geqslant 1$. *Then the sequence of vectors*

$$\mathbf{L}(\mathbf{p}_1(t)), \mathbf{L}(\mathbf{p}_2(t)), \ldots, \mathbf{L}(\mathbf{p}_m(t)), \ldots \tag{5.2}$$

converges in the metric of $L_p{}^n[a,b]$ *to* $\mathbf{r}(t) = \mathbf{L}(y(t))$, *and the sequence* (5.1) *converges uniformly on the interval* $a \leqslant t \leqslant b$ *to the unique solution* $\mathbf{y}(t)$ *of the system* (1.1) *and* (1.2).

Proof of Theorem 4. Since the solution $\mathbf{y}(t)$ of the system (1.1) and (1.2) satisfies the hypotheses of Lemma 2, for every $\epsilon > 0$ there is a vector $\mathbf{q}(t)$ of polynomials which satisfies the boundary conditions (1.2) and has the property that

$$|\mathbf{q}^{(j)}(t) - \mathbf{y}^{(j)}(t)| < \epsilon, \qquad j = 0, 1,$$

for $a \leqslant t \leqslant b$. It follows that

$$\begin{aligned}\int_a^b |\mathbf{L}(\mathbf{q}(t)) - \mathbf{r}(t)|^p\,dt &= \int_a^b |\mathbf{D}[\mathbf{q}(t) - \mathbf{y}(t)] + \mathbf{F}(t)\,[\mathbf{q}(t) - \mathbf{y}(t)]|^p\,dt \\ &\leqslant \int_a^b \{|\mathbf{D}[\mathbf{q}(t) - \mathbf{y}(t)]| + |\mathbf{F}(t)| \cdot |\mathbf{q}(t) - \mathbf{y}(t)|\}^p\,dt \\ &\leqslant \int_a^b [(F+1)\,\epsilon]^p\,dt \leqslant (b-a)(F+1)^p\,\epsilon^p,\end{aligned}$$

where F is the maximum of $|\mathbf{F}(t)|$ on the interval $a \leqslant t \leqslant b$.

Let ϵ be a given positive number and define $\mathbf{q}(t)$ as above, so that

$$\int_a^b |\mathbf{L}(\mathbf{q}(t)) - \mathbf{r}(t)|^p\,dt < \epsilon.$$

Choose M so that the components of $\mathbf{q}(t)$ are of degree at most M. Since for each m, the vector $\mathbf{p}_m(t)$ of polynomials minimizes the integral over the class of vectors of polynomials of degree at most m which satisfy the boundary conditions, when $m \geqslant M$, necessarily

$$\int_a^b |\mathbf{L}(\mathbf{p}_m(t)) - \mathbf{r}(t)|^p \, dt \leqslant \int_a^b |\mathbf{L}(\mathbf{q}(t)) - \mathbf{r}(t)|^p \, dt < \epsilon.$$

It follows that the sequence (5.2) converges in the metric of $L_p{}^n[a,b]$ to $\mathbf{r}(t) = \mathbf{L}(\mathbf{y}(t))$.

For each vector $\mathbf{p}_m(t)$, let

$$\mathbf{g}_m(t) = \mathbf{L}(\mathbf{p}_m(t)) - \mathbf{r}(t) = \mathbf{L}(\mathbf{p}_m(t) - \mathbf{y}(t)).$$

Since $\mathbf{p}_m(t)$ and $\mathbf{y}(t)$ each satisfy the boundary condition (1.2), their difference satisfies the homogeneous boundary condition

$$\mathbf{A}(\mathbf{p}_m(a) - \mathbf{y}(a)) + \mathbf{B}(\mathbf{p}_m(b) - \mathbf{y}(b)) = \mathbf{0}.$$

Let $\mathbf{G}(t,s)$ denote the Green's function for the system (1.1) and (1.2). From Theorem 1,

$$\mathbf{p}_m(t) - \mathbf{y}(t) = \int_a^b \mathbf{G}(t,s)\, \mathbf{g}_m(s) \, ds.$$

Since $\mathbf{G}(t,s)$ is continuous in the region $a \leqslant t,\ s \leqslant b$ except for a finite jump along the diagonal $t = s$, there is a constant G such that

$$|\mathbf{G}(t,s)| \leqslant G$$

for $a \leqslant t \leqslant b$ and $a \leqslant s \leqslant b$. Therefore

$$|\mathbf{p}_m(t) - \mathbf{y}(t)| \leqslant G \int_a^b |\mathbf{g}_m(t)| \, dt.$$

Since

$$\int_a^b |\mathbf{q}_m(t)| \, dt \leqslant (b-a)^{(p-1)/p} \left\{ \int_a^b |\mathbf{g}_m(t)|^p \, dt \right\}^{1/p}$$

by Hölder's inequality, it follows from the first part of the proof that

$$\int_a^b |\mathbf{g}_m(t)| \, dt$$

tends to zero as m increases. The inequality above now shows that $\mathbf{p}_m(t)$ converges uniformly to $\mathbf{y}(t)$ on the interval $a \leqslant t \leqslant b$. The theorem follows.

Additional conditions are needed to assure the convergence of the sequence of derived functions $\{\mathbf{D}\mathbf{p}_m(t)\}$. The following theorem gives a sufficient condition for the convergence of this sequence.

THEOREM 5. *Under the hypotheses of Theorem* 4, *assume that the functions*

$$\mathbf{g}_m(t) = \mathbf{L}(\mathbf{p}_m(t)) - \mathbf{r}(t), \qquad m = 1, 2, \ldots,$$

satisfy a Lipschitz condition uniformly in m on the interval $a \leqslant t \leqslant b$, i.e., *there exists a constant* λ *such that*

$$|\mathbf{g}_m(t_1) - \mathbf{g}_m(t_2)| \leqslant \lambda |t_1 - t_2|$$

whenever $a \leqslant t_1, t_2 \leqslant b$, *for* $m = 1, 2, \ldots$. *Then the sequence of derived vectors*

$$\mathbf{Dp}_1(t), \mathbf{Dp}_2(t), \ldots, \mathbf{Dp}_m(t), \ldots$$

converges uniformly on the interval $a \leqslant t \leqslant b$ *to* $\mathbf{Dy}(t)$.

Proof of Theorem 5. From the proof of Theorem 4,

$$\lim_{m \to \infty} \int_a^b |\mathbf{g}_m(t)|\, dt = 0;$$

so, from the definition of $|\mathbf{g}_m(t)|$, the components, $g_{m,i}(t)$, of $\mathbf{g}_m(t)$ have the property that

$$\lim_{m \to \infty} \int_a^b |g_{m,i}(t)|\, dt = 0, \qquad i = 1, 2, \ldots, n.$$

The definition of $|\mathbf{g}_m(t)|$ also implies that each component of $\mathbf{g}_m(t)$ satisfies a uniform Lipschitz condition on the interval $a \leqslant t \leqslant b$, so that for each $i = 1, 2, \ldots, n$, and each $m = 1, 2, \ldots$,

$$|g_{m,i}(t_1) - g_{m,i}(t_2)| \leqslant \lambda |t_1 - t_2|$$

whenever $a \leqslant t_1, t_2 \leqslant b$.

We must show that each component of $\mathbf{g}_m(t)$ converges uniformly to zero, or, equivalently, that

$$\lim_{m \to \infty} \max_{a \leqslant t \leqslant b} |g_{m,i}(t)| = 0,$$

for $i = 1, 2, \ldots, n$. We use a contrapositive argument, assuming that for some i

$$\limsup_{m \to \infty} \left\{ \max_{a \leqslant t \leqslant b} |g_{m,i}(t)| \right\} = H > 0.$$

For each $m = 1, 2, \ldots$, let t_m denote a point where the continuous function $|g_{m,i}(t)|$ attains its maximum, and let I_m be a subinterval of $[a, b]$ which has t_m as an endpoint and is of length

$$d < \min \{(b - a)/2, 2H/\lambda\}.$$

Then, when t is in I_m,

$$|g_{m,i}(t_m)| - |g_{m,i}(t)| \leqslant |g_{m,i}(t_m) - g_{m,i}(t)| \leqslant \lambda |t_m - t|,$$

so that

$$\int_a^b |g_{m,i}(t)|\, dt \geqslant \int_{I_m} |g_{m,i}(t)|\, dt$$
$$\geqslant \int_{I_m} [|g_{m,i}(t_m)| - \lambda |t_m - t|]\, dt \geqslant |g_{m,i}(t_m)|\, d - \lambda d^2/2.$$

This inequality implies that

$$\lim_{m\to\infty}\int_a^b |g_{m,i}(t)|\,dt \geqslant d(H-\lambda d/2) > 0.$$

Since

$$\lim_{m\to\infty}\int_a^b |g_{m,i}(t)|\,dt = 0,$$

it follows that each component of $\mathbf{g}_m(t)$ converges uniformly to zero, or, equivalently, that the sequence $\mathbf{L}(\mathbf{p}_m(t))$ converges uniformly to $\mathbf{r}(t)=\mathbf{L}(\mathbf{y}(t))$.

Since $\mathbf{F}(t)$ is continuous and $\mathbf{p}_m(t)$ converges uniformly to $\mathbf{y}(t)$ by Theorem 4, the product $\mathbf{F}(t)\,\mathbf{p}_m(t)$ converges uniformly to $\mathbf{F}(t)\,\mathbf{y}(t)$ on the interval $a \leqslant t \leqslant b$. Consequently,

$$\mathbf{g}_m(t) - \mathbf{F}(t)\,\mathbf{p}_m(t) + \mathbf{F}(t)\,\mathbf{y}(t) = \mathbf{D}\mathbf{p}_m(t) - \mathbf{D}\mathbf{y}(t)$$

converges uniformly to zero on the interval $a \leqslant t \leqslant b$. The conclusion follows.

6. Order of Approximation

The proof of Theorem 4 in based on Lemma 2. If $\mathbf{y}'(t)$ satisfies a Lipschitz condition, as it must if $\mathbf{F}(t)$ and $\mathbf{r}(t)$ have that property, then Lemma 3, instead of Lemma 2, can be used in that proof. In that case, we have a more specific estimate of the error $|\mathbf{y}(t) - \mathbf{p}_m(t)|$. We state this result as our final theorem.

Theorem 6. *Under the hypotheses of Theorem* 4, *assume that* $\mathbf{y}'(t)$ *satisfies a Lipschitz condition on the interval* $a \leqslant t \leqslant b$. *Then the errors* $|\mathbf{y}(t) - \mathbf{p}_m(t)|$ *have an upper bound of the order of* $1/m$.

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